Program notes

The program simulates experiments that test the claim that no local realistic representation of entangled states can match the outcome expected from quantum mechanical considerations. The default settings simulate a protocol similar to that of Kwiat et al.\textsuperscript{1} (click on Show Layout to see this), but selection of other options allows exploration of most setups from the recent literature.

To those anxious to see something, click on Run1 in the menu bar and the simulation will generate a curve showing the outcome from a local realistic model as the angle difference between polarizers is changed, using default settings for critical parameters. The program generates a population of photon pairs in the $HV/VH$ state (with $H$ as 0°) evolving to measurement at separate stations, and simulates the outcome as correlations in detection of photons are recorded at 50 random settings of polarizer 2, with polarizer 1 fixed at 0°. A different population of 1000 photons is generated for each angle difference. You will see a simple curve in the left panel that follows the QM expectation curve. However, the photons of the population have the discrete properties required of a local realistic state, so this outcome is contrary to predictions from Bell-type treatments. The right panel shows ‘singles counts’ at the four output rays at the polarization analyzers at the two measuring stations. These demonstrate that the populations tested have the stochastic distribution of $H$ and $V$ properties at the two stations appropriate to the indeterminate condition underlying theoretical treatments.

Summary of operation. The program simulates the behavior of a local, realistic (LR) population of photon pairs on analysis using polarizers and detectors at space-like separated stations. The model consists of a ‘photon source’ generating a population of correlated pairs of discrete photons, a measurement context consisting of ‘polarization analyzers’ and ‘detectors’ in separate detector channels, a choice of ‘coincidence counters’, and plotting routines. The photon source generates pairs of unit vector photons, with (in the default mode) the orientation of one rotated by 90° about the axis of propagation with respect to the other, in a reference frame with $H$ as 0°. Orientation of each pair about that axis can be fixed (‘Static’ setting), rotated through a fixed angle (‘Rotate’), or rotated at random (‘Random’ setting). Each partner in the pair is delivered, with random assignment, to one or other of a pair of ‘two-channel analyzing polarizers’, for each of which the orientation can be set with respect to the rotational frame (by default 0°). Each polarizer can be set at a fixed angle, at random rotation, to Manual selection, or to pre-programmed values (in the Run3 protocol). By default, polarizer 1 is fixed, and used as reference for the variable polarizer 2. The properties of the analyzers are idealized, so that the photon arriving at each station is conserved in either the ordinary or the extraordinary ray. The polarizers can be set to function in two different ways. In the ‘Malus LR’ setting, the distribution between rays is implemented so that the statistical fractions going to ordinary and extraordinary rays follow the law of Malus ($I = I_0 \cos^2 \theta$, where $I_0$ and $I$ are incident and transmitted intensities, and $\theta$ is the difference in orientation between photon and polarizer), the empirical behavior formalized some 200 years ago. In the ‘Bell binary’ implementation, they function so that any photon with a probability for transmission greater than 0.5 is directed to the ordinary ray and the remaining fraction appears in the extraordinary ray. The detectors are assumed to function with perfect yield. A QM simulation mode forces the photon frame to align with the polarizer frame, which would be unnatural for an LR population, but generates the QM outcome under any setting in which Malus’ law is applied.
In each cycle of the program, a selectable number of pairs of photons are generated and correlations at the outputs are then counted. In the default mode, the coincidence counters are simulated as comparators, giving an anti-correlation count, incremented for each occurrence of $Q \neq S; R \neq T; Q = T; R = S$, where $Q$, $R$, $S$, and $T$ are the values measured in the ordinary ($Q$, $S$) or extraordinary ($R$, $T$) rays, at station 1 ($Q$ and $R$), or station 2 ($S$ and $T$). Then, a pair in orthogonal alignment scores in all four cross-comparisons when the two polarizers are parallel. The protocol mimics an experiment in which a group of photon pairs is sent to distant analyzers, with the variable polarizer orientation determined during the time-of-flight of the group. The program allows selection of alternative implementations of the coincidence count, and includes gadgets to allow selection of parameters, and examination of internal results (such as photon yield, assignment of kets, polarizer angle difference, etc., and discriminator outcome), which can be tracked in any cycle. An ‘Analog mode’ option allows display of the theoretical Malus’ law yield as a function of angle difference. On selection of a Run option from the menu, the ‘experimental’ cycle is run enough times (the default of 50 random angles is sufficient to establish a set of values for polarizer difference angles that explore the full range), and plots are generated that show the correlations, and the distribution of photons between channels.

In all modes (except QM simulation mode), the behavior at the discriminators is fully compliant with locality requirements, - the photon population is established first, then ‘measurements’ are made, with those at one station completely independent of the settings of the other. For implementation of the CHSH count, an output bin with a photon in it is tagged +1 (or True in the Boolean count), while a bin with no photon is tagged –1 (or False) (the numerical representation is unimportant in the comparator counting). After measurement of all the pairs in the population, the four outcomes (as above) are then checked for correlations. The count of correlations is accumulated, and the mean extracted from a statistically significant number of photon pairs, recorded and plotted. The points displayed as dark red circles in the left panel of the program window are outcomes determined by the default comparator counts, dealt with as above. Using the same data, the program can also determine three other coincidence counts, which are displayed if selected from the ‘Left Panel’ sub-menu. These are a Boolean coincidence count, which generates points phase shifted by 90° so as to lie on the inverse sinusoid, a count following the CHSH probabilistic approach by taking cross-products between elemental outcomes at different stations, and a simulation of the Freedman and Clauser protocol (FC δ-count) (see below). All parameters can be examined in the Gadgets. The elemental coincidence scores of the CHSH count (2 or -2), and the anti-correlation or the coincidence counts (0 or 4) (see text boxes labeled $QS + RS + RT - QT$, Anti-correlation and Coincidence) always show one or other value. The mean outcome values reflect the weight in distribution of elemental values, dependent on settings. (Note that these counts reflect measurement at four detectors (at ordinary and extraordinary rays of polarization analyzers at both stations, as recommended2), and analysis of all four possible pairwise cross correlations. However, with perfect detectors, half these counts are redundant. The Gadgets values reflect the full count, appropriate for a scaling to ±2 or 4, but scaling in the left panel display is selectable, and the default setting (giving $2\cos^2\sigma$ amplitude) reflects the non-redundant count.)

Functions requiring further explanation are the Controls, Displays, Menu items, and various Gadgets, which allow examination of internal values, ‘state preparation’ through insertion of additional refractive elements, and display of additional information.

Main Menu items
**File** allows the user to select a filename, and to store there (in text format), the x- and y-values resulting from the last **Run**, as displayed in the left-hand panel. By default, these are in tab-separated columns, and store values corresponding to the data points plotted in the Left Panel, as selected in the Left Panel menu. The columns are in the following order: 1) Angle difference; 2) anti-correlation count; 3) coincidence count; 4) CHSH count; 5) delta function; 6) FC δ-count. In the present implementation, only the last set of the results of a **Run3** are available, so export of data with fixed values for polarizer difference angle would have to be via successive **Run1** simulations in **Manual** mode.

**Run1** initiates an ‘experiment’ by calling Make Light, Measurement, and Plot Points routines (see below) at each of \( n \) polarizer settings, with \( n \) set using the **No. of angles** slider. Each photon pair of the population is measured using the current Polarizer mode, with polarizer settings as determined by Polarizer orientation settings (the angle difference, which changes as each cycle is run if the **Manual** option is not selected, is shown in the text box). The outcome values are stored, and plotted (using the Plot curves subroutine) before a new cycle is initiated with a different set of photon pairs, measured with a new angle difference. Each cycle generates a single point in the left panel.

**Run2** initiates a similar set of ‘measurements’, except that the population of photon pairs is not changed before the angle difference is reset. The results are essentially the same in both modes, but coherence patterns in distribution sometimes appear in the right panel, so **Run1** is more ‘realistic’ for a real population of photon pairs. **Run2** is particularly useful in analytical analysis (using the **Analog** option) of the Malus’ law behavior when a single photon pair is used, because the behavior of a single pair with a particular (randomly set) allocation of the nominally \( H \) or \( V \) member to each channel can be examined as a function of the angle difference between polarizers.

**Run3** initiates a succession of **Run1** experiments with polarizer 1 fixed at the angle set in the Polarizer orientation control, and the angle of polarizer 2 determined under program control. The angle is set initially to give an angle difference of -90°, then incremented by 7.5° (chosen so as to include values at canonical angle difference) over the range -90° to 90°. This is equivalent to running each angle in Manual Polarizer mode. However, under **Run3**, instead of a single run at each angle difference, a mean score is accumulated at each angle from a number of runs, and the average values are used to generate the coincidence counts plotted in the left panel. This improves the signal/noise ratio substantially. The **No. for average** slider is used to set how many runs will be averaged; a value in the range 5 – 10 provides a useful compromise between signal/noise and time taken.

The following three menu items are the guts of the program:

**Make Light** invokes the subroutine that generates a photon population.

**Measurement** invokes the subroutine that simulates the measurement context.

**Plot Points** invokes the subroutine that plots a point in the left panel.

Each subroutine can be invoked separately by clicking on the menu item, and the results can be explored through the Gadgets.

The **Run** options invoke these subroutines in a For…Next loop for a succession of different settings of angle difference as appropriate to choices from the Controls.
Clear clears the two display panels, and also clears the arrays used to store intermediate values (used for the text displays in the Gadgets in the upper section of the window).

Unit Circle activates display of a separate window in which the vectors representing the two photons (shown as yellow vectors spanning both quadrants) and two polarizers are shown in a unit circle. With polarizer 1 and $H$ photons at 0°, the diagonal lines represent orientations of polarizer 2, which by reference to polarizer one as a zero reference, give vectors for polarizer angle difference, $\sigma$. Vector projections from polarizer 2 on the photon vectors are also displayed. In the present version, only the case in which photon orientation ($H = 0^\circ$) is aligned with polarizer vector 1 set at 0°. This represents the projections expected from the orthodox QM treatment. The cursor can be used to select points in the plotted surface. Points of interest are the interceptions of the diagonals on the circumference, which yield $\cos \sigma$ and $\sin \sigma$ values which project onto the photon vectors to provide the dot products. The values returned are in the labeled text boxes at the top. The squares of these values are the Malus’ law yields (bottom), and showing the partitioning into ordinary and extraordinary rays.

In the Unit Circle window menu, Clear erases the display, and Quit closes the window.

Show Layout opens a separate window with a picture of the experimental layout simulated.

Help brings up this document in a separate window.

Quit exits the program.

Controls

The controls determine the rotational properties of the photon population, the Bell-state or source of photons, the state preparation of the photon population, the mode of operation of the polarizers and the settings of the polarizers.

The properties of the photon population are set up through selection of Photon orientation and Photon mode. The Photon orientation can be selected as either ‘Static’ or ‘Random’, or rotated through a fixed angle (‘Rotate’, including a ‘Show/Hide’ button). The Photon mode can be one of two different Bell states, or as two distinct populations. When one of the Bell states is selected, a population of pairs of photons is generated in which the properties reflect the settings, and orthogonal correlations are implemented by random allocation on generation of each pair (for example any pair might be nominally either $HV$ or $VH$, either with $H$ as zero in the orientation of the photon reference frame and $V$ at 90° (‘Static’), or with the same orthogonal difference but rotated about the axis of propagation by a fixed angle (‘Rotate’), or at stochastic orientation (isotropic in the plane of measurement) (‘Random’). These controls are discussed in greater detail next.

Photon orientation can be selected from the following:

Static mode. In ‘Static’ mode, each photon of a pair is either horizontally ($H$, angle $0^\circ$) or vertically polarized ($V$), but the allocation of $HV$ and $VH$ is at random. This simulates the output on selection from intersection points of the circles generated when an interference filter has selected monochromatic populations from the two cones output after parametric down conversion (PDC) with a BBO crystal.
Random mode. In ‘Random’ mode, the photons of the pair are orthogonally correlated (at 90° for HV/VH pairs), but the pair is rotated at random about the axis of propagation over the full range up to 360°, so that in the population, each photon pair has a different random orientation in this range. With HH/VV pairs, this generates a population that approximates the “Ca-cascade” output used in early experiments.

Rotate mode. The orientation of a static pair is rotated about the axis of propagation by the angle shown in the associated text box. The angle can be changed by using the Rotation Angle slider below this control.

Show/Hide button. This activates the Rotate mode, but hides the value for angle, and deactivates the Gadgets. Since you don’t know the angle, this allows a guessing game in which you have to figure out what angle for polarizer 1 is needed to get a full-visibility curve.

Photon mode.

The Photon mode selection box can be used to imitate the two main Bell-states, HV/VH and HH/VV. An additional option, ‘PDC EO’, allows all photons in the ordinary cone from a source emulating a PDC output to be sent to one station, and all in the extraordinary cone to the other.

In either Bell state mode, photons of a pair are sent to the two stations at random, with an overall equipartition of HV/VH (or HH/VV) orientations. The vectorial properties of the photon pairs are determined by the Photon orientation mode and by the photon reference frame; the orientation angle for each photon used in implementation of the discriminator function is given by the reference frame (the angle of the H photon), offset according to orthogonal designation (0° for H or 90° for V) in Static mode, and any additional rotation in Random or Rotate modes.

Note that the Photon mode determines the state of the photon source before ‘state preparation’, as can be examined in the Photon angle text boxes in the Gadgets.

HV/VH mode. This is the default option, appropriate to simulation of parametric down conversion (PDC) in a BBO-crystal under type II conditions, with populations selected from the intersection of cones as in1. On selection of Static mode, the reference frame is 0°, the default HV/VH option.

HH/VV mode. This generates pairs with photons in the same orientation, but with an equal mix of H and V photons in each channel. This option can be used to simulate an atomic-cascade if the photon orientation is set to Random, which distributes the kets at random while conserving the pairwise correlation (the orientation of the reference frame for each pair is randomly set).

PDC EO option allows exploration of the outcome from a type II phase-matching PDC source if all extraordinary ray photons (H) were to be sent to one detector (station 1), and all ordinary ray photons (V) to the other (station 2). This option is not a classical entangled state, because, although the photons are pair-wise correlated, there is no ambiguity in the orthogonal allocation. By insertion of HWP 2 at 45°, the state (HH/HH) can be set to simulate a fully polarized source, with both photons reaching the detectors in the same orientation (this would be the case if a Ca-cascade output was polarized). Similarly, insertion of HWP 1 set at 45° will generate a VV/VV state.

In the first two options, the orientation of a pair about the axis of propagation can be static, random, or rotated, as discussed under the Photon Source controls. The Random
setting is not implemented under PDC E O setting; it would generate a population indistinguishable from the $HH/VV$ setting with Random photons.

The orientation of the photon population can be rotated either by using the Rotate option, or by toggling insertion of HWP 0 into both beams. The overall rotation of the beam is additive, but with the rotation due to HWP 0 at double the angle set in the slider. The Photon pairs slider determines the number of photons pairs generated in a population.

**State preparation** controls. The default “state preparation” built into the simulation emulates the mixing of $H$ and $V$ populations by selection of from the intercepts of the two cones output on PDC (see1), to generate a population of $HV/VH$ pairs. Other populations can be selected by choice of Photon mode. These can be reconfigured to give different Bell states by insertion of half-wave plates (HWP) in either beam.

**HWP simulation.** Insertion of a HWP can be toggled by clicking the appropriate check box. The HWP function is implemented by rotating the vector of photon population by $2\lambda$, where the value for $\lambda$ is set in the HWP rotation slider, or by writing in the associated text box.

**HWP 0.** ‘HWP 0’ is inserted in both rays to allow rotation of the photon frame. It can also be used to simulate the initial state preparation suggested in1 to change $H$ to $V$ and vice versa (when set at 45°), used experimentally to allow correction of time-tags.

**HWP 1.** The ‘HWP 1’ option simulates insertion of a half-wave plate in channel 1 in front of polarizer 1. The function can be toggled on or off by clicking on the check box. The default is with angle set at 45° (the classical HWP function, rotating the beam by 90°), but the rotation angle can be set to any angle using the slider to rotate the beam.

**HWP 2.** Similarly, the ‘HWP 2’ option inserts a plate in channel 2. The HWP 1 or HWP 2 options allow exploration of the different Bell states. (See also ‘PDC E O’ option in the Photon mode control). In fitting theoretical lines to the points generated, the theoretical green curves are calculated assuming this classical behavior of the HWPs.

**Polarizer settings.** Three different options can be selected for the polarizer orientations:

**Fixed** mode. When ‘Fixed’ is selected, angle differences are set so that polarizer 1 is fixed at an angle (with reference to horizontal at 0° in the common reference frame) determined by the setting of the slider labeled P1 or typing into the associated text box (initially 0°), and polarizer 2 (P2) is rotated to a random angle in the range ±180° for each cycle of measurement. The default setting for polarizer 1 is horizontal.

**Random** mode. When ‘Random’ is selected, polarizer 1 is rotated at random through the range ±45°, and polarizer 2 is rotated at random through the range ±180°. The No. of angles slider determines the number of values for polarizer angle difference, and hence the number of cycles, in a Run1 (or Run2).

**Manual** mode. The orientations of the two polarizers can be set using the sliders (or text boxes) to the right, and the angle values are displayed in the associated text boxes. An angle of 0° is (by default) horizontal. In Manual mode, the number of cycles (and the
value in No. of angles slider) is set to 1, and Run1 elicits a single cycle at fixed values of polarizer angles. This is the same as clicking successively on Make Light, Measurement and Plot Points. In either case, the pairwise data values can be read out in the Gadgets. Manual mode is also called to implement the Run3 mode in which a selectable number of Runs are averaged. The No. for average slider sets how many Run1 cycles are averaged at each setting of Polarizer 2. The angle of polarizer 2 is changed under program control (fixed increments of 7.5° over a range for angle difference of -90° to 90°). This mode provides a good comparison with the standard protocols in which the angles are preset, and improves the quality of the simulation by including more values in the mean than a single Run1.

Polarizer mode controls. The discriminator function can be set either to implement the law of Malus (‘Malus LR’ mode), or to operate in a scalar fashion (‘Bell binary’ mode). In both cases, the behavior is simulated on the basis of the law of Malus yield, calculated as below for each photon at each polarizer. Values are stored internally, can be examined in the gadgets, and are used to calculate the Analog points. However, in the simulation, they are used only to determine probabilities for distribution of photons to ordinary and extraordinary rays by stochastic allocation.

Malus LR mode. In the ‘Malus LR’ setting, Malus’ law is implemented statistically, by comparing the Malus’ law yield to a random number between 0 and 1; if the Malus’ law value is greater, the photon is sent to the ordinary ray, so that when applied to a statistically significant number of photons, the (statistical) fraction of photons appropriate to Malus’ law expectations appears there, and the remaining fraction in the extraordinary ray.

Bell binary mode. In the ‘Bell binary’ setting, the discrimination is such that any photon with a probability for transmission (based on Malus’ law) greater than 0.5 is directed to the ordinary ray, otherwise to the extraordinary ray. In effect, an H (0°) photon will pass if the polarizer orientation (with respect to the same reference frame) is in a horizontal quadrant (for example, 45° to -45°, or the opposite quadrant), or a V (90°) photon if the polarizer orientation is in a vertical quadrant (45° to 135°, etc.).

QM simulation. The rotational invariance predicted by QM treatments depends on an alignment of the photon and polarizer frames implemented in the measurement context. In Shimony’s description, this happens when a photon at indeterminable orientation has its orientation “actualized” on encountering a polarizer. The alignment occurs “…by substituting the transmission axis of analyzer I for x and the direction perpendicular to both z and this transmission axis for y…” with x and y representing the orthogonal allocation (0, 1 or H, V) designated in the wavefunction equation and reflected in operation of the Pauli matrices. For an LR population, in which photons have real vectors, such functionality at the polarizer would be unnatural. However, the behavior can be readily simulated with an LR population by using appropriate “If … then … else” statements, as long as the photons are labeled to designate a nominal type (H or V) (see source code for implementation). In effect, on reaching the measurement station, the photons are rotated from their real orientation so as to align with the polarizer frame so
that the $H$ photons have the same orientation as the fixed polarizer, the $V$ photons have the orthogonal orientation. To achieve the expected outcome, the measurement context has to differentiate $H$ from $V$, and information about the rotational frame actualized at station 1 has to be made available at station 2 to implement a complementary function there. This alignment is trivial in silico but unnatural under local realistic predicates. The effect is to generate the same outcome as Shimony’s QM treatment, rotationally invariant and with full-visibility.

Display panels

Left panel
The coincidence count as a function of the difference between the polarizer angles is plotted in the left-hand panel. By default, this is an anti-correlation count, on a scale 0 - 2. For the $HV/VH$ photon population, the maximum is found when the two polarizers are aligned, and minimum when they are orthogonal. With the $HH/VV$ population, the curves are phase shifted by 90° (maximal when polarizers are orthogonal in the anti-correlation count). The display for angle difference is limited to one hemisphere (-90° to 90°) (values outside the range are adjusted by symmetry rotation).

All counts are implemented on each cycle. The Left Panel menu item allows selection of display for different coincidence counts, and this selection is also implemented in the Save as function of the File menu.

Two options allow the display to be configured so as to emphasize different points.

- **Symbol.** This provides a drop-down list from which different symbols can be selected for plotting the next and subsequent sets of data points. Note that the color-coding according to coincidence count will not be changed, but filled symbols will be filled, and open symbols will carry the appropriate line color.

- **Scale.** Measurement in all four detectors (as in the simulation) is preferred experimentally, because it maximizes coincidence counts with inefficient detectors. When implemented in the simulation, this gives an outcome scaled to four units. However, the scaling is unnatural because, with perfect detectors, half the coincidences are redundant. The Scale option (initially set to 0 - 2) allows the user to choose what scale to use in displaying the curves generated. With the initial setting, the Run functions generate curves scaled to 2 units (the natural outcome for a single curve from the two non-redundant coincidence measurements). Choice of scale 0 - 1 would be appropriate if measurements scored only 1 coincidence. A choice of scale to 4 units is sometimes convenient in comparison with the inequalities, where the limits ≤2 and ≤2.83 relate to a $4\cos^2\sigma$ curve.

  Choice of any range displays the same data; only the normalization of curve and scale is changed. The scaling to 2 units is natural, and the scaling to 4 units implies summation of two curves (in effect a rescaling from 2 to 4 units).

  (See **Note on scaling**… below for further discussion.)
**Left Panel sub-menu**

The three different coincidence counts all give essentially the same curve, but are phase-shifted depending on count type. The default (Anti-correlation, red symbols) is an anti-correlation count, but if you select CHSH count (gray symbols) or Coincidence (green symbols) from the Left Panel menu, the curve is inverted (phase-shifted by 90° compared to the default). Curves from all counts selected can be displayed. Both coincidence counts generate an output in the range 0 – 4, appropriate to the left scale. The CHSH algorithm generates an outcome in the range ±2 (see Notes on scaling below), appropriate to the right scale. Selection of different Photon mode options is discussed above, but note in the present context that, because of the difference in orthogonal correlation, when the HH/VV option is selected, the outputs are also phase-shifted by 90° with respect to the HV/VH curves. Other options are detailed under **Left Panel Options** below.

**Theoretical curves**

**Aligned LR (LR0).** The dark red curve shows the full-visibility result expected from an aligned LR model (also from QM models). The amplitude is $n(\cos^2\sigma)$, where $n$ is the Scale selected (for example, $2(\cos^2\sigma)$ when Scale 0-2, the default, is selected), and $\sigma$ is the difference in angle between the two polarizers. When photon and polarizer frames are aligned as expected in the QM treatment, the curve can be derived directly from the analog curves (Malus’ law expectations) generated (see **Analog yield** under Right panel below).

**Zigzag curves (LR1).** The diagonal straight lines show the result expected from Bell’s binary ‘local realistic’ model. For an $HV/VH$ population using an anti-correlation count scaled to $n$, the curves follow $n \times (\frac{|2\sigma|}{180})$ (with $\sigma$ in degrees), which is the inverse of Bell’s equation.

**Half-visibility curves (LR2).** When photon orientation is ‘Random’, the statistical counts in ‘Malus LR’ mode follow a half-visibility curve, and the scale of the theoretical curves is adjusted to $0.5n(\cos^2\sigma)$ where $n$ is the Scale chosen. This gives the analytical outcome suggested in 4.

**FC curves.** If the Freedman-Clauser counting protocol is selected (FC δ-count in Left panel menu), the amplitudes generated from an isotropic source, or an ordered source with equal partitioning of $H$ and $V$ photons, are half the visibility of the other counts, because the coincidence yields at a station with polarizer in place are normalized to the yield in the absence of a polarizer at the other station. At a station with polarizer in place, only half the photons pass through. However, with a polarized source (PDC E O), the curves show full visibility because, at appropriate alignment, all the photons can pass through the polarizer. The theoretical curves are adjusted accordingly.

**Malus’ law curves (green curves).** The green curves are generated by calculation of the joint probabilities (coincidences) expected when pairs of photons in orthogonal correlation are partitioned by the polarization analyzers at the two stations. They are calculated by cross multiplication of the Malus’ law yields, $I$, at individual stations, given for $HV/VH$ pairs by $I_{Q_1}^{HV}$, $I_{Q_2}^{HV}$, $I_{R_1}^{HV}$, $I_{R_2}^{HV}$, for $HV$ pairs at outputs $Q, S, R, T$, respectively, with similar terms for $VH$ pairs, or for $HH/VV$ states. Cross products are summed following the same pattern as in eq. 2, (for
example $I_{01}^{HV} \times I_{02}^{HV}$ for QS with HV pairs) but with sign appropriate to settings. However, this requires that both sets of orthogonal possibilities (for example HV and VH) are considered (see Note on scaling below). When the photon and polarizer frames are aligned, this generates the same outcome as shown by the red curve ($n\cos^2\sigma$ when scaled to 0 - $n$). However, the green curves can fit the points generated by all LR treatments in the simulation, including those generated when the photon orientation is not aligned with a fixed polarizer. Then, curves of reduced visibility, with the maximum phase-shifted from the $n\cos^2\sigma$ maximum by the difference between the polarizer and photon frames. The visibility goes to zero when the difference is $\pm 45^\circ$. The properties of the curve depend on whether HV/VH or HH/VV pairs are involved, and on implementation of different combinations of HWPs in state preparation. Display of the green curves is implemented if the photon and polarizer frames are different whenever Malus LR operation is selected with a Static photon population. By selection of Malus’ law curves in the Left Panel sub-menu, they can also be plotted for the aligned case instead of the default QM curves.

A special case is activated by selection of ‘Plot envelope’ from the Left Panel menu when the photon orientation is random. This generates an envelope of green curves. Each curve represents the curve that would be generated on measurement at all angle differences for a random setting for photon pair orientation and a fixed value of polarizer 1 orientation. The random value for photon orientation is set using the orientation of the first photon pair of the current population. (Note that no envelope is generated if Run2 is used.)

**QM simulation** (blue curve and symbols) - When this option is selected (see notes under Polarizer controls above), a blue full-visibility QM curve is plotted to fit the points generated.

**Right panel mode** allows a choice of what will be displayed in the Right Panel.

**Show Singles.** By selection from the drop-down menu, the mean of the number of hits in each ray (singles counts) are displayed (polarizer 1, ordinary ray (Q), red; extraordinary ray (R), violet; polarizer 2, ordinary ray (S), orange; extraordinary ray (T), green).

**Analog Yield.** The ‘Analog Yield’ option allows display of the analog (fractional) yield difference between the Malus’ law values for the outputs of the two polarizers as a function of angle difference. The analog values are derived from the difference between the photon yield values shown in the gadgets. With Analog selected, and depending on selection of different items from the Right Panel submenu, different theoretical lines in the right panel show the Malus’ law yield differences in each hemisphere for ordinary and extraordinary rays. The allocation of points and curves depends on the Bell-state chosen.

For an HV/VH aligned population, the theoretical curve (dark red) shown in the left panel is $4\cos^2\sigma$ (or $2\cos^2\sigma$ when Scale is 0-2), given by summation of differences between Malus’ law yields as follows: $\Delta I_{HV}^0 = \cos^2\sigma$ (dark blue points and curve), and
\( \Delta I_{VH}^O = -\cos^2\sigma \), (light blue); and complementary curves \( \Delta I_{HV}^E = \sin^2\sigma \) and \( \Delta I_{VH}^E = -\sin^2\sigma \), (red and yellow, respectively) for the extraordinary rays. Recognizing that \( \sin^2x = 1 - \cos^2x \), the \( 4\cos^2\sigma \) curve then comes from

\[
S_{LR} = (\Delta I_{HV}^O - \Delta I_{VH}^O) - (\Delta I_{HV}^E - \Delta I_{VH}^E)
\]

With \( HH/VV \) pairs, the yield differences are phase shift by 90\(^\circ\), so that the dark blue points follow the red curve (\( \sin^2\sigma \)), the red points follow the dark blue curve (\( \cos^2\sigma \)), the light blue points follow the orange curve (\( -\sin^2\sigma \)), and the orange points follow the light blue curve (\( -\cos^2\sigma \)).

The Analog curves are useful in dissecting the more esoteric effects when photon and polarizer frames are misaligned, or scrambled.

**Right Panel** sub-menu - the choices change depending on option selected. The default option is to display the Single counts, - the mean of sums of yields in individual rays (Q, R, S, or T, see layout). If Analog Yield mode has been selected, the theoretical curves and calculated values for fractional (analog) yield differences in the ordinary and extraordinary rays at the two stations (see Analog option for description and equations). By default, only the values in the ordinary rays are displayed, since the values in the extraordinary rays are complementary, but it can be informative to see all.

**Panel cursor X, Y.** These text boxes (bottom right of control panel) display the position of the mouse pointer when the mouse is clicked on either of the display panels, allowing a read-out of parameters (left text box for Left Panel, right for Right Panel).

**Gadgets**

The gadgets occupy the top right corner of the main window. They allow examination of the results of the last set of measurements, and can be used to test the routines of the program. The slider labeled Photon pair index sets the pointer to arrays containing stored data, and the information is displayed in the different text boxes as the pointer changes. After a Run1 or Run2, the data stored are for the last cycle at the Angle Difference shown. If inspection of an individual cycle is required, click successively on Make Light, Measurement, and Plot points to mimic a cycle. The different text boxes return the property indicated by the label:

**Photon pair index** The slider sets pointer for the arrays of stored data and information, for which values are displayed in the different text boxes. The arrays store values for parameters as follows:

- **Photon Angle** The actual orientation (with respect to the 0\(^\circ\) horizontal reference) of each photon as it arrives at the polarizers (the allocation is as indicated by the Ket distribution text boxes). The values indicate the vector after addition of any rotation to the nominally \( H \) and \( V \) photons, but before ‘State preparation’ by HWP1 or 2.
- **Polarizer Angle** The orientation of each polarizer (with respect to the horizontal reference 0\(^\circ\)).
- **Photon Yield** The Malus’ law value, determined from the difference in angle between the photon electrical vector and the polarizer orientation,
as shown in the two preceding gadgets, for each of the two polarizers.

**Ket distribution**  The (random) allocation of the nominally horizontal photon to either of the measurement stations (0 = not allocated, 1 = allocated; in the text boxes, 1 0 is nominally \(HV\) pair, 0 1 is nominally \(VH\) pair).

**Angle difference**  The difference between the angles of the two polarizers, with polarizer 1 as reference.

**Detector Response**  Shows the allocation of photons to the ordinary (d1o, d2o) and extraordinary (d1e, d2e) bins for the two detectors (1 shows a photon, -1 no photon), based in the discriminator function selected, and the Malus’ Law value. These are the values of \(Q\) (d1o), \(R\) (d1e), \(S\) (d2o), and \(T\) (d2e) after detection.

**Pairwise counts**  The value for the elemental CHSH count of \((QS + RS + RT – QT)\) is shown for each photon pair (always +2 or -2). Below this, the text boxes labeled Anti-correlation and Coincidence show the elemental pairwise outcome of anti-correlation (the default count) and Boolean coincidence counts (always 0 or 4). As you click through the information for each pair of a photon population using the Photon pair index slider, the elemental count values flip, but show a distribution with a probability reflecting the Malus’ law values. The CHSH count is similar to the BCHSH\(^{2,3}\)/Leggett\(^5\) count (see below).

**Left Panel sub-menu options**

**Anti-correlation count**  (dark red symbols) - the default count, incremented by 1 for each of the following conditions: \(Q \neq S; R \neq T; Q = T; R = S\) (eq. 1). The mean value is plotted.

**CHSH counts**  (gray symbols) – the mean of elementary values for the outcome as angle difference is varied, given by eq. 2

\[
2 + \left(\sum_{i=1}^{n} (QS + RS + RT – QT)_i\right) / n \quad (eq. 2)
\]

the 2 being an offset added to bring the range of values (+2) to the same scale as the other plots (0-4). This expression gives the QM curve as long as the law of Malus is implemented, but the LR curve if discriminators are assumed to behave in a binary fashion. When selected, a scale appropriate to the function without offset is displayed on the right-hand scale. See Notes on scaling... below for relation to BCHSH count.

**Coincidence count**  (green symbols) – this parameter uses Boolean logic to simulate electronic gates following the algorithm below. This generates a mean coincidence count (the inverse of the anti-correlation count) through eq. 3:

\[
\overline{C \mathcal{L}} = \left(\sum_{i=1}^{n} -(q \text{ XNOR } s + r \text{ XNOR } t + q \text{ XOR } t + r \text{ XOR } s)_i\right) / n
\]

Here, \(q, r, s,\) and \(t\) are the Boolean observables (True if a photon was detected, otherwise False) at the ordinary and extraordinary rays of stations A and B for the \(i^{th}\) pair of a
population of \( n \) pairs measured at a particular angle difference. The Truth Table values for these functions are shown below, and always return 0 or 1:

<table>
<thead>
<tr>
<th>Input A</th>
<th>Input B</th>
<th>Output</th>
<th>Input A</th>
<th>Input B</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

By swapping the XNOR and XOR operators, the anti-correlation count can be generated, and the treatment is then formally equivalent to the default count. All three counts scale to 4 in a range dependent on elemental values.

**“Ideal” \( \delta \)-function** (magenta symbols) - this function displays the Bell inequality as a function of angle difference, \( \sigma \), represented by the difference between the experimental points and Bell’s LR linear zigzag. The experimental points are those given by the “CHSH count” function, using eq. 2, and the linear zigzag is given by \((4 \times |2\sigma|/180)\). Then the outcome \{(eq. 2) + 2\} - \((4 \times |2\sigma|/180)\} is adjusted to midscale, and provides a visual representation of the “inequalities” over the full range of angle difference. The function is similar in spirit to the \( \delta \) function suggested by Freedman and Clauser\(^6\), but scaled to full visibility. Note that if the photon source and polarizer frames are not aligned, or if the polarizer behavior applied is not Malus’ law compliant, this treatment will generate a plot that is more confusing than helpful!

**QM intercepts** - when selected, this function shows the intercepts on ordinate and abscissa of points on the curve corresponding to the canonical QM values, - amplitudes at 22.5\(^o\) and 67.5\(^o\), for which the difference on the sinusoidal curve scaled to 4 units is the QM expectation value of 2.83 (or 1.414 if scaled to 2 units).

**FC \( \delta \)-count** (cyan symbols) - the coincidence count is implemented so as to emulate the procedure suggested by Freedman and Clauser\(^6\) to generate the elemental counts \( R_1, R_2, R_0 \), and \( R(\sigma) \), used to normalize the experimental data to QM expectations through:

\[
\frac{R(\sigma)}{R_0} = 0.25(\epsilon_M^1 + \epsilon_m^1)(\epsilon_M^2 + \epsilon_m^2) + 0.25(\epsilon_M^1 - \epsilon_m^1)(\epsilon_M^2 - \epsilon_m^2)F_0\cos2\sigma \tag{eq. 4}
\]

With either a stochastic source, or with Bell-states generating an equal mix of \( H \) and \( V \) at each station, to give a symmetrical distribution, this protocol generates an outcome scaled to 0.5. The half-visibility is a consequence of the isotropic/symmetrical source. The first term on the right is the cross-product from \( R_1/R_0 \) and \( R_2/R_0 \) terms, each with value 0.5, from measurement of yields at a polarizer (at any orientation) with reference to those in the absence of a polarizer at the other station (half the photons are lost at the polarizer). The second term scales the sinusoidal outcome curve in the range 0 - 0.25.

It is important to note that the program does not implement eq. 4, but that the pairwise coincidences accumulate to generate the outcome implicit in the equation.

The curves are displayed with respect to the full-visibility of a QM-expectation at the scale chosen (\( n = 1 \) for coincidences counted only in the ordinary rays), so that with an isotropic/symmetrical source, the FC \( \delta \)-count curves have half the amplitude of the
other coincidence counts. With a polarized source of photons pairs with the same orientation (essentially the configuration expected when photons are actualized under QM predicates), the simulation generates full visibility curves, which match those from the other coincidence counts in this scale, because $R_{\sigma}/R_{0}$ values then fall in the range 0 - 1. Such a polarized source can be simulated by choice of ‘PDC EO’ is the photon source, and insertion of HWP2 (at 45°) to rotate the $V$ ray by 90°, making this source $HH/HH$. A curve with the same full-visibility, but phase shifted by 90°, is obtained if HWP2 is omitted.

Malus’ law curves - when selected, the full-visibility theoretical curves generated with aligned frames are replaced by their Malus’ law equivalents (the green curves).

Note on scaling of the expectation curves

The scale of outcome curve at any particular setting of the fixed polarizer depends on the number of coincidences tested at the elemental level. For $n$ comparisons, the natural scaling is an $n\cos^2\sigma$ curve, defining the maximal visibility for the aligned case. In early experiments, measurements were made only in the ordinary rays, and since information is restricted to instances in which a photon is detected, with ideal detectors $n = 1$ for each pairwise measurement, or a natural $\cos^2\sigma$ curve. Current protocols often improve the statistics by measurement in both ordinary and extraordinary rays, and would then give a natural $2\cos^2\sigma$ curve. In principle, measurement in both rays allows four comparisons (cross-products $QS, QT, RS$ and $RT$), each of which can register a coincidence, giving a maximum of $n = 4$ from the set above, and a scale of 4 units. However, half these counts are redundant (the information can be inferred from the detected photons and conservation laws); the same scaling is achieved in the QM count by normalization.

The canonical QM outcome depends on polarizer settings, $\alpha, \beta, \alpha', \beta'$, and involves four separate experiments with comparisons $\alpha, \beta; \alpha', \beta'; \alpha, \beta'$; $\alpha', \beta'$. In line with the above, the four measurements would give a $4\cos^2\sigma$ curve outcome curve defining the maximal amplitude. The QM limit of $\leq 2.83$ comes from this scaling, and choice of values for $\beta$ and $\beta'$ to set $\sigma$ at 22.5 and 67.5, with the reference polarizer at either $\alpha = 45^\circ$ or at $\alpha' = 0^\circ$. Then, outcome values ($\pm \cos^2\sigma$) from each of the four measurements are $\pm 0.7071$, which, on summation with appropriate sign, gives 2.83. However, this limit value comes from an expectation that the photon frame will become aligned with that of the fixed polarizer. If aligned, the same full-visibility curve is expected at any setting of the fixed polarizer, or if the polarizer is fixed at a particular value, for any orientation of the photon pairs, including a stochastic population. Since the curve at any particular fixed polarizer setting can, in principle provide the same information as that from any other setting of the fixed polarizer, a single curve is sufficient to provide the information needed to calculate the limit. Although, with a single setting, only two of the above four coincidences ($\alpha, \beta$ and $\alpha', \beta'$ for example) would be counted, the QM expectations can be obtained simply by rescaling that curve by 2 to give a $4\cos^2\sigma$ curve, effectively equivalent to summing two identical curves with different settings, $\alpha = 45^\circ$ or at $\alpha' = 0^\circ$, for example. A curve in either scale could be considered to show a full-visibility. However, because the expectation values require four measurements, only the $4\cos^2\sigma$ curve provides values matching the expectations on substitution of the canonical values (22.5° and 67.5°) for $\sigma$. 
The three algorithms for coincidence counts in the program generate curves scaled internally to the amplitude range of 0-4 (or ±2). This scale depends on a count of four coincidences (from cross-products $QS$, $RS$, $RT$, and $QT$) at the 4 detectors in ordinary and extraordinary rays at both stations (this configuration would reflect the experimentalist’s preferred choice), and on perfect optical elements. However, with detector efficiency at 100%, failure to detect a photon carries the same informational weight as detection, and if the coincidences from these are included, half the counts are then redundant. The natural scale, allowing measurement at two values for $\sigma$ referred to the same reference, is a $2\cos^2\sigma$ curve, and this is the default display scale used in the simulation. As noted above, under QM predicates, the $4\cos^2\sigma$ curve can be obtained either by summation of two curves, or simply by rescaling. The same choices are available under LR constraints, but in that case, the alignment of photon frames has to be explicit, rather than implicit as under QM predicates (i.e. the photon frame has to be rotated to match the orientation of the fixed polarizer).

In the conventional CHSH LR framework\textsuperscript{2,3,5,7}, in line with spin quantum numbers, a photon emerging from the polarization analyzer was assigned a value of +1 if in the ordinary ray, or -1 if in the extraordinary ray, and the cross-products from photon pairs also have these values. On substitution into the standard equation,

\[ S_{\text{elemental}} = E(\alpha, \beta) + E(\alpha', \beta) + E(\alpha', \beta') - E(\alpha, \beta'), \]

the outcome takes values +2 or -2, and this sets the amplitude of the mean to a scale of 4 in the range ±2. In the simulation, coincidences are scored in four rather than two comparisons, between detectors at Q, R, S and T. If the conventional CHSH scoring by the assignment of ±1 values is used, attempts to calculate coincidences through cross-products of elemental probabilities fails because half the elemental terms are undefined. To get over this problem, a different assignment is used in the simulation; if a photon is detected, an elemental outcome is assigned a value +1, if no photon is detected, -1, and use of this information means that all correlations can be scored directly. This results in a pair-wise outcome count of ±2, and hence a mean count constrained to the same range as CHSH, giving the same limit of ≤2. These values are handled essentially through eq. 2, in which $S_{LR}$ is the mean, and the terms of the inner bracket are cross products of elemental ±1 values. At a particular of the polarizers for $\sigma$:

\[ S_{LR}^\sigma = \left( \sum_{i=1}^{n} (QS + RS + RT - QT) \right) / n \]

This equation scrupulously follows the standard LR constraints. However, the outcome represents the same double-counting as noted above, so half of these are redundant. When scaled to the natural 2 units, my CHSH count and Shimony’s BCHSH count both give curves on the same $2\cos^2\sigma$ scale (display is offset by 2 in the Figures to match the scale of other counts). To show full visibility (the $4\cos^2\sigma$ curve), both require alignment of photon and polarizer frames, and a count of four coincidences, two at each setting of the fixed polarizer (or the summation of two curves). In the BCHSH treatment, this is handled by implicit alignment under QM predicates. To simulate this under LR constraints, an explicit realignment of the photon frame is required. A rescaling of either of the 2-unit curves would represent summation of two curves after implicit (QM) or explicit (LR) realignment of the photon frame to match the reorientation of the fixed polarizer.

References


